FINITE AND INFINITE SERIES OF THE RECIPROCAL OF POLYNOMIAL FACTORIALS ARISING FROM DIFFERENCE OPERATOR

G.Britto Antony Xavier[a], S.Gokulakrishnan[b] and V.Chandrasekar [a,b]

[a]Department of Mathematics,
Sacred Heart College,
Tirupattur - 635601, Tamil Nadu, S.India,
[b,a,b] Department of Mathematics
SKP Engineering College,
Tiruvannamalai - 606 611,
Tamil Nadu, S.India.
[a]E-Mail: shcbritto@yahoo.co.in

Abstract

In this paper, we derive the formula for finding the value of sum of $(m-1)$ time partial sums of the reciprocal of product of polynomial factorials in the field of finite difference methods. Suitable examples are provided to illustrate the main results.

Key words: Generalized difference operator, Polynomial factorial, Infinite series.

AMS Subject classification: 39A10, 39A11, 39A13.

1. Introduction

The Fractional Calculus is currently a very important research field in several different areas: physics (including classical and quantum mechanics and thermodynamics), chemistry, biology, economics and control theory [7, 12, 13, 14, 15]. In 1989, K.S.Miller and Ross [11] introduced the discrete analogue of the Riemann-Liouville fractional derivative and proved some properties of the fractional difference operator. The main definition of fractional difference equation (as done in [11]) is the ν^{th} fractional sum of $f(t)$ by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} \frac{\Gamma(t-s)}{\Gamma(t-s-(\nu-1))} f(s), \tag{1}$$

where $\nu > 0$. On the other hand fractional h sum of order $m \geq 1$ $((\Delta_h^{-m} f)(t)$, Definition 2.8 of [10]) is very useful to derive many interesting results in a different way in finite difference methods such as the sum of the m^{th} partial sums on n^{th} powers of arithmetic, arithmetic-geometric progressions and products of n consecutive terms of arithmetic progression using $\Delta_\ell^{-m} u(k)$ [9,10]. During the last decades several fractional sums for various functions have been investigated by numerous mathematicians (cf. e.g, [2,3,5,6] and the bibliography quoted there).

In the existing literature, there are several series in which certain series have direct formula to find its value and some series have no direct formula. For example, $\sum_{n=1}^{k} \frac{1}{n^p}$ and its partial sums etc., and the m-series defined below.

Let $\ell > 0$ and $u(k)$ be real valued function on $[0, \infty)$ and $u(k) = 0$ for all $k \in (-\infty, 0)$. Then, for $m \in \mathcal{N}(1)$, the m-series of $u(k)$ with respect to ℓ is defined as below:

$$1-series; \quad u_{1(\ell)}(k)=u(k-\ell)+u(k-2\ell)+...+u\left(k-\left[\frac{k}{\ell}\right]\ell\right),$$

$$2-series; \quad u_{2(\ell)}(k)=u_{1(\ell)}(k-\ell)+u_{1(\ell)}(k-2\ell)+...+u_{1(\ell)}\left(k-\left[\frac{k}{\ell}\right]\ell\right),$$

and in general, $m-$ series;

$$u_m(\ell)(k)=u_{(m-1)(\ell)}(k-\ell)+u_{(m-1)(\ell)}(k-2\ell)+...+u_{(m-1)(\ell)}\left(k-\left[\frac{k}{\ell}\right]\ell\right).$$

If we take $u(k), u(k-\ell), \cdots, u(k-\left[\frac{k}{\ell}\right]\ell)$ are the amounts of infection of disease at the time $k, k-\ell, \cdots, k-\left[\frac{k}{\ell}\right]\ell$ respectively in living things, then $u_{m(\ell)}(k+m\ell)$ gives the total amount of infection of the disease for m-generations. In the field of Health Science, it is necessary to find the exact value of m-series for the proper treatment of medicine.

We find that the m-series of $u(k)$ with respect to ℓ is the numerical solution of the difference equation

$$\Delta_\ell^m v(k)=u(k),\ k\in[0,\infty),\ \ell>0, \tag{2}$$

and the complete solution of (2) provide the value of the m-series. Hence in this paper, we derive formula for m-series to product polynomial and polynomial factorials by numerical-complete solution of the difference equation (2).

2. Preliminaries

Before stating and proving our results, we present some notations, basic definitions and preliminary results which will be used in the subsequent discussions.

Let $[x]$= integer part of x, $\ell\in(0,\infty)$ be fixed real, $k\geq 0$ be variable, $j=k-\left[\frac{k}{\ell}\right]\ell$ be starting value for k with respect to ℓ, $J_m=\{1,2,\dots,m\}$, $0(J_m)=\{\phi\}$, ϕ is empty set, $1(J_m)=\{\{1\},\{2\},\cdots,\{m\}\}$,

$2(J_m)=\{\{1,2\},\{1,3\},\cdots,\{1,m\},\{2,3\},\cdots,\{2,m\},\cdots,\{m-2,m-1\}\}$. In general, $t(J_m)$ is the set of all subsets of size t, arranging in ascending order, from the set J_m, $\wp(J_m)=\bigcup_{t=0}^{m}t(J_m)$ is the power set of J_m, $\sum_{t=1}^{m}f(t)=0$ for $m<1$ and $\prod_{i=2}^{t}f(i)=1$ for $t\leq 1$, $u_i(k)=\Delta_\ell^{-1}u_{i-1}(k)|_{(i-1)\ell+j}^{k}=\Delta_\ell^{-1}u_{i-1}(k)-\Delta_\ell^{-1}u_{i-1}((i-2)\ell+j)$ for $i=2,\cdots,m$, $u_1(k)=\Delta_\ell^{-1}u(k)$ and $u_0(k)=u(k)$.

Definition 2.1. [8] For a real valued function $u(k)$, the generalized difference operator Δ_ℓ and its inverse on $u(k)$ are respectively defined as

$$\Delta_\ell u(k)=u(k+\ell)-u(k),\ k\in[0,\infty),\ \ell\in(0,\infty), \tag{3}$$

and

$$\text{if } \Delta_\ell v(k)=u(k),\ \text{then } v(k)\ =\ \Delta_\ell^{-1}u(k)+c_j. \tag{4}$$

Where C_j is constant for all $k\in N_\ell(j)$

ISBN 978-93-82338-91-8 © 2013 Bonfring

In general if a function $v(k)$ satisfies the difference equation (2) then it is called a solution of the difference equation (2).

Lemma 2.2. [8] If s_r^n and S_r^n are the Stirling numbers of the first and second kinds respectively, and $k_\ell^{(n)} = k(k-\ell)(k-2\ell)\cdots(k-(n-1)\ell)$, then

$$k_\ell^{(n)} = \sum_{r=1}^{n} s_r^n \ell^{n-r} k^r, \quad k^n = \sum_{r=1}^{n} S_r^n \ell^{n-r} k_\ell^{(r)} \text{ and } \Delta_\ell^{-1} k_\ell^{(\nu)} = \frac{k_\ell^{(\nu+1)}}{(\nu+1)\ell}. \tag{5}$$

Lemma 2.3. *[8]* Let $u(k)$, $k \in [0,\infty)$ be real valued function. Then for $k \in [\ell,\infty)$

$$\Delta_\ell^{-1} u(k)\Big|_j^k = \sum_{r=1}^{[\frac{k}{\ell}]} u(k-r\ell). \tag{6}$$

Theorem 2.4. *[4]* If $\lim_{k\to\infty} \Delta_\ell^{-r} u(k) = 0$ for $r = 1,2,\cdots,m$ and $k \in [m\ell,\infty)$, then

$$\Delta_\ell^{-m} u(k)\Big|_k^\infty = \sum_{r=m}^{\infty} \frac{(r-1)^{(m-1)}}{(m-1)!} u(k-m\ell+r\ell). \tag{7}$$

Corollary 2.5. *[4]* Let $k \in [0,\infty)$ and $\lim_{k\to\infty} \Delta_\ell^{-1} u(k) = 0$. Then

$$\Delta_\ell^{-1} u(k)\Big|_k^\infty = \sum_{r=1}^{\infty} u(k-\ell+r\ell). \tag{8}$$

3. Main Results

In this section , we derive the product of Reciprocal of polynomial factorial and find the sum of partial sum of finite and infinite series on reciprocal of arithmetic progression by using the inverse of generalized difference operator.

Theorem 3.1. ***(Generalized Product Formula)***

If n is positive integer, $u_i(k)$, $i = 1,2,\cdots,n$ are real valued functions, then

$$\Delta_\ell \prod_{i=1}^{n} u_i(k) = \prod_{i=1}^{n-1} u_i(k)\Delta_\ell u_n(k) + \sum_{t=0}^{n-2} \prod_{s=n-t}^{n} \prod_{i=1}^{n-t-2} u_s(k+\ell) u_i(k) \Delta_\ell u_{n-t-1}(k). \tag{9}$$

Proof. From the Definition 2.1, we have

$$\Delta_\ell[u_1(k)u_2(k)\cdots u_n(k)] = u_1(k+\ell)u_2(k+\ell)\cdots u_n(k+\ell) - u_1(k)u_2(k)\cdots u_n(k). \tag{10}$$

Adding and subtracting terms $\prod_{i=1}^{n-j} u_i(k)v_i(k)$, $j = 2,3,\cdots,(n-1)$, we obtain

$$\Delta_\ell[u_1(k)u_2(k)\cdots u_n(k)] = \prod_{i=1}^{n-1} u_i(k)\Delta\ell u_n(k) + u_n(k+\ell)\prod_{i=1}^{n-2} u_i(k)\Delta_\ell u_{n-1}(k)$$

$$+u_n(k+\ell)u_{n-1}(k+\ell)\prod_{i=1}^{n-3} u_i(k)\Delta_\ell u_{n-2}(k) + u_n(k+\ell)u_{n-1}(k+\ell)u_{n-2}(k+\ell)$$

$$\prod_{i=1}^{n-4} u_i(k)\Delta_\ell u_{n-3}(k) + \cdots + u_n(k+\ell)u_{n-1}(k+\ell)\cdots u_3(k+\ell)u_1(k)\Delta_\ell u_2(k)$$

$$+u_n(k+\ell)u_{n-1}(k+\ell)\cdots u_3(k+\ell)u_2(k+\ell)\Delta_\ell u_1(k). \quad (11)$$

The proof follows by (10) and (11).

Example 3.2. In (9), by taking $n = 4$, $u_i(k) = \frac{1}{(k+t_i\ell)_\ell^{m_i}}$, for $i = 1, 2, 3, 4$, we have

$$\Delta_\ell \left[\prod_{i=1}^{4} \frac{1}{(k+t_i\ell)_\ell^{m_i}}\right] = \sum_{r=0}^{3} \frac{-\ell}{\prod_{i=1}^{4-r}(k+\ell+t_{5-i}\ell)_\ell^{m_{5-i}+1}\prod_{p=1}^{r}(k+t_p\ell)_\ell^{m_p}}.$$

Theorem 3.3. Let m and n be the positive integers. Then,

$$\Delta_\ell^{-m}[\prod_{i=1}^{n} u_i(k)] = \prod_{i=1}^{n-1} u_i(k)\Delta_\ell^{-m}u_n(k) + \sum_{t=1}^{m}\left\{\Delta_\ell^{-1}u_n(k+\ell)\Delta_\ell[\prod_{i=1}^{n-1} u_i(k)]\right\}. \quad (12)$$

Proof. From Definition 2.1, we have

$$\Delta_\ell^{-1}\left[\prod_{i=1}^{n-1} u_i(k)\Delta_\ell w_n(k)\right] = \prod_{i=1}^{n-1} u_i(k)w_n(k) -$$

$$\Delta_\ell^{-1}\left[\sum_{t=0}^{n-2}\prod_{s=n-t}^{n-1}\prod_{i=1}^{n-t-1} u_s(k+\ell)w_n(k+\ell)u_i(k)w_n(k)\Delta_\ell u_{n-t-1}(k)\right]. \quad (13)$$

(12) follows by substituting $u_n(k) = \Delta_\ell w_n(k)$ in (13).

Corollary 3.4. If n is positive integer, then

$$\Delta_\ell^{-1}[\prod_{i=1}^{n} u_i(k)] = \prod_{i=1}^{n-1} u_i(k)\Delta_\ell^{-1}u_n(k) - \Delta_\ell^{-1}[\Delta_\ell^{-1}u_n(k+\ell)(\Delta_\ell\prod_{i=1}^{n-1} u_i(k))]. \quad (14)$$

Proof. The proof follows by substituting $m = 1$ in (12).

Theorem 3.5. If m is positive integer and $r_n \geq (m+1) + \sum_{i=1}^{n-1} r_i$, then

$$\Delta_\ell^{-m}\left[\frac{\prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}}{(k+t_n\ell)_\ell^{(r_n)}}\right] = \prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}\Delta_\ell^{-m}\frac{1}{(k+t_n\ell)_\ell^{(r_n)}}$$

$$+\sum_{t=1}^{m}\left\{\Delta_\ell^{-1}\frac{1}{(k+t_n\ell)_\ell^{(r_n)}}\Delta_\ell[\prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}]\right\}. \quad (15)$$

Proof. Proof follows by taking $u_i(k) = (k+t_i\ell)_\ell^{(r_i)}, i = 1, 2, \cdots, n-1$ and $u_n(k) = \frac{1}{(k+t_n\ell)_\ell^{(r_n)}}$ in (12).

Theorem 3.6. If m is positive integer and $r_n \geq (m+1) + \sum_{i=1}^{n-1} r_i$, then

$$\sum_{r=m}^{[\frac{k}{\ell}]}\left[\frac{\prod_{i=1}^{n-1}(k+(t_i-r)\ell)_\ell^{(r_i)}}{(k+(t_n-r)\ell)_\ell^{(r_n)}}\right] = \prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}\Delta_\ell^{-m}\frac{1}{(k+t_n\ell)_\ell^{(r_n)}}$$

$$+\sum_{t=1}^{m}\left\{\Delta_\ell^{-1}\frac{1}{(k+t_n\ell)_\ell^{(r_n)}}\Delta_\ell[\prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}]\right\}. \quad (16)$$

Corollary 3.7. If $\ell \in (0,\infty)$, then

$$\Delta_\ell^{-1}\left[\frac{(k+t_1\ell)_\ell^{(2)}(k+t_2\ell)_\ell^{(2)}}{(k+t_3\ell)_\ell^{(6)}}\right] =$$

$$-\frac{1}{5\ell}\frac{(k+t_1\ell)_\ell^{(2)}(k+t_2\ell)_\ell^{(2)}}{(k-\ell+t_3\ell)_\ell^{(5)}}+\frac{2}{5}\Delta_\ell^{-1}\left[\frac{(k+t_2\ell)_\ell^{(1)}(k+t_1\ell)_\ell^{(2)}}{(k+t_3\ell)_\ell^{(5)}}\right]$$

$$+\frac{2}{5}\Delta_\ell^{-1}\left[\frac{(k+t_1\ell)_\ell^{(1)}(k+t_2\ell)_\ell^{(2)}}{(k+t_3\ell)_\ell^{(5)}}\right]. \quad (17)$$

Proof. The proof follows by substituting $m = 1$, $n = 3$, $r_1 = r_2 = 2$ and $r_3 = 6$ in (15).

Theorem 3.8. Let $k \in [0,\infty)$ and $j = k - \left[\frac{k}{\ell}\right]\ell$. Then,

$$\sum_{r=1}^{\left[\frac{k}{\ell}\right]}\frac{(k+(t_1-r)\ell)_\ell^{(2)}(k+(t_2-r)\ell)_\ell^{(2)}}{(k+(t_3-r)\ell)_\ell^{(6)}} =$$

$$-\frac{1}{5\ell}\frac{(k+t_1\ell)_\ell^{(2)}(k+t_2\ell)_\ell^{(2)}}{(k-\ell+t_3\ell)_\ell^{(5)}}+\frac{2}{5}\Delta_\ell^{-1}\left[\frac{(k+t_2\ell)_\ell^{(1)}(k+t_1\ell)_\ell^{(2)}}{(k+t_3\ell)_\ell^{(5)}}\right]$$

$$+\frac{2}{5}\Delta_\ell^{-1}\left[\frac{(k+t_1\ell)_\ell^{(1)}(k+t_2\ell)_\ell^{(2)}}{(k+t_3\ell)_\ell^{(5)}}\right]\Big|_j^k \quad (18)$$

Proof. The proof follows from (6) and (17).

Corollary 3.9. Let $k \in [0,\infty)$ and $j = k - \left[\frac{k}{\ell}\right]\ell$. Then,

$$\sum_{r=2}^{\left[\frac{k}{\ell}\right]}(r-1)\frac{(k+t_1\ell-r\ell)_\ell^{(1)}(k+t_2\ell-r\ell)_\ell^{(1)}}{(k+t_3\ell-r\ell)_\ell^{(5)}} = \frac{1}{12\ell^2}\frac{(k+t_1\ell)_\ell^{(1)}(k+t_2\ell)_\ell^{(1)}}{(k-2\ell+t_3\ell)_\ell^{(3)}}$$
$$+\frac{1}{4}\sum_{t=1}^{2}\frac{(-1)^t}{3^{2-t}}\Delta_\ell^{-t}\left[\sum_{i=1}^{2}(k+(i-1)\ell+t_i\ell)_\ell^{(1)}\frac{1}{(k-2\ell+t_3\ell+i\ell)_\ell^{(3)}}\right]\Big|_{\ell+j}^{k} \quad (19)$$

Proof. The proof follows by substituting $m = 2$in (16) and (6).

The following example is illustrates Corollary 3.9.

Example 3.10. In (19), substituting $k = 47, \ell = 3, j = 2, t_1 = 4, t_2 = 5, t_3 = 6$, we get

$$\sum_{r=2}^{15}\left[(r-1)\frac{(59-2r)_2^{(1)}(62-2r)_2^{(1)}}{(65-2r)_2^{(5)}}\right] = \frac{1}{108}\frac{(59)_2^{(1)}(62)_2^{(1)}}{(59)_2^{(3)}}$$
$$+\frac{1}{4}\sum_{t=1}^{2}\frac{(-1)^t}{3^{2-t}}\Delta_2^{-t}\left[\sum_{i=1}^{2}(47+(i-1)2+2t_i)_2^{(1)}\frac{1}{(59+2i)_2^{(3)}}\right]\Big|_5^{47} = 0.0165389$$

The following example is the illustration of Theorem 3.8.

Example 3.11. In (18), substituting $k = 63, \ell = 4, j = 3, t_1 = 3, t_2 = 4, t_3 = 5$, we get

$$\sum_{r=1}^{15}\left[\frac{(75-4r)_4^{(2)}(79-4r)_4^{(2)}}{(83-4r)_4^{(6)}}\right] = -\frac{1}{20}\frac{(75)_4^{(2)}(79)_4^{(2)}}{(79)_4^{(5)}}$$
$$+\frac{2}{5}\Delta_\ell^{-1}\left[\frac{(75)_4^{(2)}(79)_4^{(1)}}{(83)_4^{(5)}}\right]+\frac{2}{5}\Delta_\ell^{-1}\left[\frac{(75)_4^{(1)}(79)_4^{(2)}}{(83)_4^{(5)}}\right] = 0.055070622$$

Theorem 3.12. If m is positive integer and $r_n \geq (m+1) + \sum_{i=1}^{n-1} r_i$, then

$$\sum_{r=m}^{\infty}\left[\prod_{i=1}^{n-1}(k+(t_i+r-1)\ell)_\ell^{(r_i)}\frac{1}{(k+(t_n+r-1)\ell)_\ell^{(r_n)}}\right] = \prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}$$

$$\Delta_\ell^{-m}\frac{1}{(k+t_n\ell)_\ell^{(r_n)}} + \sum_{t=1}^{m}\left\{\Delta_\ell^{-t}\frac{1}{(k+t_n\ell)_\ell^{(r_n)}}\Delta_\ell[\prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}]\right\}\Big|_k^\infty \qquad (20)$$

Proof. The proof follows from (7) and (15).

Corollary 3.13. If $r_n \geq 2 + \sum_{i=1}^{n-1} r_i$, then

$$\sum_{r=1}^{\infty}\left[\prod_{i=1}^{n-1}(k+(t_i+r-1)\ell)_\ell^{(r_i)}\frac{1}{(k+(t_n+r-1)\ell)_\ell^{(r_n)}}\right] = \prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}$$

$$\left(\frac{-1}{(r_n-1)\ell(k+t_n\ell)_\ell^{(r_n-1)}}\right) - \frac{1}{(r_n-1)\ell(k+t_n\ell)_\ell^{(r_n)}}\Delta_\ell[\prod_{i=1}^{n-1}(k+t_i\ell)_\ell^{(r_i)}]\Big|_k^\infty \qquad (21)$$

Proof. The proof follows by substituting $m = 1$ in (20).

Corollary 3.14. If $r_n \geq 3 + \sum_{i=1}^{n-1} r_i$, then

$$\sum_{r=2}^{\infty}\left[(r-1)\frac{(k+t_1\ell+(r-2)\ell)_\ell^{(1)}(k+t_2\ell+(r-2)\ell)_\ell^{(3)}}{(k+t_3\ell+(r-2)\ell)_\ell^{(7)}}\right] = \frac{-1}{30\ell^2}$$
$$\frac{(k+t_1\ell)_\ell^{(1)}(k+t_2\ell)_\ell^{(3)}}{(k-2\ell+t_3\ell)_\ell^{(5)}} + \frac{1}{10\ell}\Delta_\ell^{-1}\left[\frac{(k+t_1\ell)_\ell^{(1)}(k+t_2\ell)_\ell^{(2)}}{(k-\ell+t_3\ell)_\ell^{(5)}}\right] + \frac{1}{30\ell}\Delta_\ell^{-1}\left[\frac{(k+\ell+t_2\ell)_\ell^{(3)}}{(k-\ell+t_3\ell)_\ell^{(5)}}\right] \qquad (22)$$
$$-\frac{1}{2\ell}\Delta_\ell^{-2}\left[\frac{(k+t_1\ell)_\ell^{(1)}(k+t_2\ell)_\ell^{(2)}}{(k+t_3\ell)_\ell^{(6)}}\right] - \frac{1}{6\ell}\Delta_\ell^{-2}\left[\frac{(k+\ell+t_2\ell)_\ell^{(3)}}{(k+t_3\ell)_\ell^{(6)}}\right].$$

Proof. Substituting $m = 2$ and $n = 3$ in (20), we obtain

$$\sum_{r=2}^{\infty}\left[\frac{(k+(t_1+r-2)\ell)_\ell^{(r_1)}(k+(t_2+r-2)\ell)_\ell^{(r_2)}}{(k+(t_3+r-2)\ell)_\ell^{(r_3)}}\right] =$$

$$\frac{(k+t_1\ell)_\ell^{(r_1)}(k+t_1\ell)_\ell^{(r_2)}}{(r_3-1)^{(2)}\ell^2(k+t_3\ell)_\ell^{(r_3)}} + \sum_{t=1}^{2}\left\{\Delta_\ell^{-t}\frac{1}{(k+t_3\ell)_\ell^{(r_3)}}\Delta_\ell[\prod_{i=1}^{2}(k+t_i\ell)_\ell^{(r_i)}]\right\}\Big|_k^\infty \qquad (23)$$

(22) follows by taking $r_1 = 1, r_2 = 3$ and $r_3 = 7$ in (23).

The following example is illustrate Corollary 3.14.

Example 3.15. In (22), taking $k = 58, \ell = 5, t_1 = 6, t_2 = 7, t_3 = 8$, we obtain

$$\sum_{r=2}^{\infty}\left[(r-1)\frac{(78+5r)_5^{(1)}(83+5r)_5^{(3)}}{(88+5r)_5^{(7)}}\right] = \frac{1}{750}\frac{(88)_5^{(1)}(93)_5^{(3)}}{(88)_5^{(5)}}$$
$$-\frac{1}{50}\Delta_\ell^{-1}\left[\frac{(88)_5^{(1)}(93)_5^{(2)}}{(93)_5^{(5)}}\right] - \frac{1}{150}\Delta_\ell^{-1}\left[\frac{(98)_5^{(3)}}{(93)_5^{(5)}}\right]$$
$$+\frac{1}{10}\Delta_\ell^{-2}\left[\frac{(88)_5^{(1)}(93)_5^{(2)}}{(98)_5^{(6)}}\right] + \frac{1}{30}\Delta_\ell^{-2}\left[\frac{(88)_5^{(3)}}{(98)_5^{(6)}}\right] = 0.0000620$$

Example 3.16. In (21), by taking $n = 3$, $r_1 = 1, r_2 = 2$ and $r_3 = 7$, we find

$$\sum_{r=1}^{\infty}\left[\frac{(k+t_1\ell+(r-1)\ell)_\ell^{(1)}(k+t_2\ell+(r-1)\ell)_\ell^{(2)}}{(k+t_3\ell+(r-1)\ell)_\ell^{(6)}}\right]=$$
$$-\frac{1}{5\ell}\frac{(k+t_1\ell)_\ell^{(1)}(k+t_2\ell)_\ell^{(2)}}{(k-\ell+t_3\ell)_\ell^{(5)}}-\frac{1}{20\ell}\frac{(k+\ell+t_2\ell)_\ell^{(2)}}{(k-\ell+t_3\ell)_\ell^{(4)}}-\frac{1}{30\ell}\frac{(k+t_1\ell)_\ell^{(1)}}{(k-\ell+t_3\ell)_\ell^{(3)}}$$
$$-\frac{1}{30\ell}\frac{(k+\ell+t_2\ell)_\ell^{(1)}}{(k-\ell+t_3\ell)_\ell^{(3)}}-\frac{1}{30\ell}\frac{(k+\ell+t_2\ell)_\ell^{(1)}}{(k-\ell+t_3\ell)_\ell^{(3)}}-\frac{1}{60\ell}\frac{1}{(k-\ell+t_3\ell)_\ell^{(2)}}$$
$$-\frac{1}{60\ell}\frac{1}{(k-\ell+t_3\ell)_\ell^{(2)}}-\frac{1}{60\ell}\frac{1}{(k-\ell+t_3\ell)_\ell^{(2)}}-\frac{1}{10\ell}\frac{(k+t_1\ell)_\ell^{(1)}(k+t_2\ell)_\ell^{(2)}}{(k-\ell+t_3\ell)_\ell^{(4)}}.$$

In particular $k=43,\ell=2,t_1=2,t_2=3$ and $t_3=4$, we obtain

$$\sum_{r=1}^{\infty}\left[\frac{(45+2r)_2^{(1)}(47+2r)_2^{(2)}}{(49+2r)_2^{(6)}}\right]=-\frac{1}{10}\frac{(47)_2^{(1)}(49)_2^{(2)}}{(49)_2^{(5)}}$$
$$-\frac{1}{40}\frac{(51)_1^{(2)}}{(46)_2^{(4)}}-\frac{1}{60}\frac{(47)_2^{(1)}}{(49)_2^{(3)}}-\frac{2}{60}\frac{(51)_2^{(1)}}{(49)_2^{(3)}}$$
$$-\frac{3}{120}\frac{1}{(49)_2^{(2)}}-\frac{1}{20}\frac{(47)_2^{(1)}(49)_2^{(2)}}{(49)_2^{(4)}}=0.000143413867$$

References

[1] R.P. Agarwal, Difference Equations and inequalities , Marcel Dekker, New york,2000.

[2] R.Almeida, D.F.M.Torres, *Necessary and Sufficients for the Fractional Calculus of Variations with Caputo Derivatives*, Commun. Non linear Sci. Numer. Simul., 16(2011), No.3, 1490 - 1500. arXiv:1007.2937.

[3] F.M.Atici and P.W.Eloe, *A Transform Method in Discrete Fractional Calculus*, International Journal of Difference Equations, 2(2) (2007), 165 - 176.

[4] G. Britto Antony Xavier, V. Chandrasekar, S. U. Vasanthakumar, and B. Govindan, *Discrete Gamma (Factorial) Function and Its Series in Terms of a Generalized Difference Operator*, Advances in Numerical Analysis, vol. 2012, Article ID 780646, 13 pages, 2012.doi:10.1155/2012/780646.

[5] N.R.O. Bastas, R.A.C. Ferreria and D.F.M.Torres, *Necessary Optimality Conditions for Fractional Difference Problems of the Calculus of Variations*, Discrete and Continuous Dynamical Systems Series B (DCDS - B), (accepted 2010).

[6] N.R.O. Bastas, R.A.C. Ferreria and D.F.M.Torres, *Discrete time fractional variational problems*, Signal Process, 91(2011), No.3, 513 - 524. arXiv:1005.0252.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[8] M.Maria Susai Manuel, G.Britto Antony Xavier and E.Thandapani, *Theory of Generalized Difference Operator and Its Applications*, Far East Journal of Mathematical Sciences, 20(2) (2006), 163 - 171.

[9] M.Maria Susai Manuel, G.Britto Antony Xavier, V.Chandrasekar and R.Pugalarasu, *Theory and Application of the Generalized Difference Operator of the n^{th} Kind (Part - I)*, Demonstratio Mathematica, 45(1) (2012), 95 - 106.

[10] M.Maria Susai Manuel, V.Chandrasekar and G.Britto Antony Xavier, *Some Applications of the Generalized Difference Operator of the n^{th} Kind*, Far East Journal of Applied Mathematics, 66(2) (2012), 107 - 126.

[11] K.S.Miller,B.Ross, *Fractional difference calculus, in "Univalent functions,fractional calculus, and the applications (Koriyama, 1988)"*, 139-152, Horwood, Chichester, 1989.

[12] K.S.Miller,B.Ross, *An introduction to the fractional calculus and fractional differential equations*, wiley & Sons, New York,1993.

[13] M. D. Ortigueira, *Fractional central differences and derivatives*, J. Vib. Control 14 (2008), No. 9-10, 1255 - 1266.

[14] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

[15] M. F. Silva, J. A. Tenreiro Machado and R. S. Barbosa, *Using fractional derivatives in joint control of hexapod robots*, J. Vib. Control 14(2008), No. 9-10, 1473-1485.

[16] Saber. N. Elayadi, An introduction to Difference Equation, 2nd edition, Springer, 1999.

[17] Walter. G. Kelley and Allan C. Peterson, Difference Equations : An Introduction with Applications, Academic press,Inc., 1991.